Quantum Morphisms

Lecture 10

Review

$\boxed{(M,b)\text{-game}}$

$M \in \mathbb{Z}_2^{m\times n}$, $b \in \mathbb{Z}_2^m$; $S_\ell = \{ i \in [n] : M_{\ell i} = 1 \}$

Alice + Bob sent $\ell, k \in [m]$, respond with $f : S_\ell \to \mathbb{Z}_2$, $f' : S_k \to \mathbb{Z}_2$

Win if $\sum_{i\in S_\ell} f(i) = b_\ell$, $\sum_{j \in S_k} f'(j) = b_k$, + $f(i) = f'(i)$ $\forall i \in S_\ell \cap S_k$.

Multiplicative form of $Mx=b$:

$$\sum_{i\in S_\ell} x_i = b_\ell \longrightarrow \prod_{i \in S_\ell} x_i = (-1)^{b_\ell}$$

$$x_1 + x_2 + x_3 = 1 \longrightarrow x_1 x_2 x_3 = -1$$

$$a \in \mathbb{Z}_2 \longrightarrow (-1)^a \in \{\pm 1\}$$

Quantum solution to $Mx=b$: $A_i \in B(\mathcal{H})$ for $i \in [n]$ s.t.

1) $A_i^* = A_i$ + $A_i^2 = I$ $\forall i \in [n]$;
2) $A_i A_j = A_j A_i$ if $\exists\, \ell \in [m]$ s.t. $i, j \in S_\ell$;
3) $\prod_{i\in S_\ell} A_i = (-1)^{b_\ell} I$ $\forall \ell \in [m]$.

Solution group $\Gamma(M,b)$: generators g_i for $i\in[n]$ and J satisfying the relations:

1) $g_i^2 = e \;\; \forall i\in[n]$ and $J^2 = e$;
2) $g_iJ = Jg_i \;\; \forall i\in[n]$;
3) $g_ig_j = g_jg_i$ if $\exists\, \ell\in[m]$ s.t. $i,j\in S_\ell$;
4) $\prod_{i\in S_\ell} g_i = J^{b_\ell} \;\; \forall\, \ell\in[m]$.

Theorem (Cleve, Liu, Slofstra): TFAE:

1) the (M,b)-game has a perfect qc-strategy;
2) $Mx=b$ has a quantum solution;
3) the solution group $\Gamma(M,b)$ has $J\neq e$.

Theorem (Cleve & Mittal): TFAE:

1) the (M,b)-game has a perfect q-strategy;
2) $Mx=b$ has a finite dimensional quantum solution;
3) the solution group $\Gamma(M,b)$ has a finite dimensional representation ϕ s.t. $\phi(J)\neq\phi(e)$.

Theorem: TFAE:

1) the (M,b)-game has a perfect classical strategy;
2) $Mx=b$ has a classical solution;

2') $Mx=b$ has a 1-dimensional quantum solution;

2'') $Mx=b$ has a commutative quantum solution;

3) the abelianization of the solution group $\Gamma(M,b)$ has $J \neq e$.

Slofstra's Embedding Theorem

Tsirelson's problem & embedding thm for groups

Let Γ' be a finitely presented group, let J' be a central element of Γ', and let $w_1, \dots, w_n \in \Gamma'$ be s.t. $w_i^2 = e$ $\forall i \in [n]$. Then there is a BLS $Mx=b$, distinct indices $i_1, \dots, i_n$ and an embedding (injective hom.) $\phi: \Gamma' \to \Gamma(M,b)$ s.t. $\phi(J') = J$ and $\phi(w_i) = g_{i_k}$ $\forall k \in [n]$.

Corollary 1: There is a BLS $Mx=b$ s.t. the (M,b)-game has a perfect qc-strategy but no perfect q-strategy

Strengthening (Slofstra): There is a BLS $Mx=b$ s.t. the (M,b)-game can be won with probability arbitrarily close to 1 using q-strategies, but has no perfect q-strategy.

$\Rightarrow$ The set of q-correlations is not closed.

Corollary 2: It is undecidable to determine if a BLS game has a perfect qc-strategy.

Later (Slofstra): It is undecidable to determine if:

1) a BLS game has a perfect q-strategy;
2) a BLS game can be won with probability arbitrarily close to 1 using q-strategies.

$C_x(m_A, m_B, n_A, n_B)$

C_x = set of x-correlations, $x \in \{loc, q, qs, qa, qc, ns\}$

loc - classical

q - finite dimensional tensor-product framework

qs - ∞-dimensional tensor-product framework

qa - closure of q = closure of qs

qc - commuting operator framework

ns - non-signalling

Before Slofstra:

$$C_{loc} \subsetneq C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc} \subsetneq C_{ns}$$

After Slofstra:

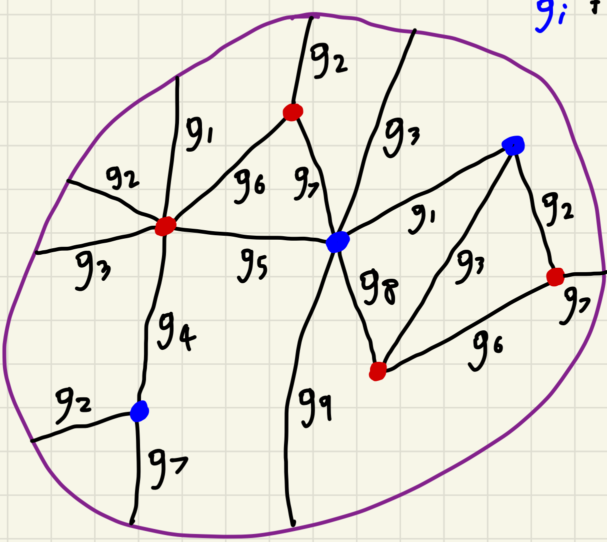
$$C_{loc} \subsetneq C_q \subseteq C_{qs} \subsetneq C_{qa} \subseteq C_{qc} \subsetneq C_{ns}$$

Now:

$$C_{loc} \subsetneq C_q \subsetneq C_{qs} \subsetneq C_{qa} \subsetneq C_{qc} \subsetneq C_{ns}$$

Slofstra's proof uses:

g_i for $i \in [n]$, J-generators of $\Gamma(M,b)$



● = J relations

● = e relations

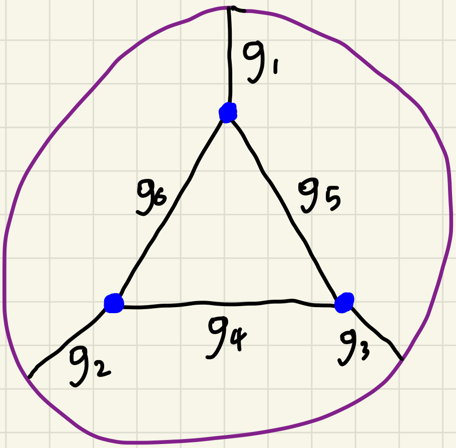
E.g. here we have

$g_2 g_6 g_7 = J$ & $g_1 g_2 g_3 = e$

as relations (among others).

Product of generators incident to boundary in counter-clockwise order $= J^{\# \bullet\text{'s}}$: $g_2 g_1 g_2 g_3 g_2 g_7 g_9 g_7 g_3 = J^4 = e$

Smaller example:



$g_1 g_6 g_5 = e$

$g_2 g_4 g_6 = e$

$g_3 g_5 g_4 = e$

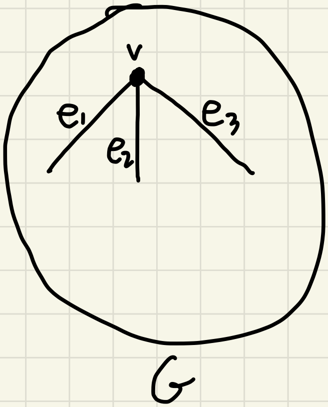
$\Rightarrow g_1 g_2 g_3 = e$

Constructing $Mx=b$ with q-solution but no classical solution

Let G be a graph. The incidence matrix of G is the $V(G) \times E(G)$ matrix M s.t.

$$M_{v,e} = \begin{cases} 1 & \text{if } v \text{ is an endpoint of } e \\ 0 & \text{o.w.} \end{cases}$$

We consider systems $Mx=b$ where M is the incidence matrix of a graph G and $b \in \mathbb{Z}_2^{V(G)}$. Thus we view the edges of G as our variables and its vertices as our equations.



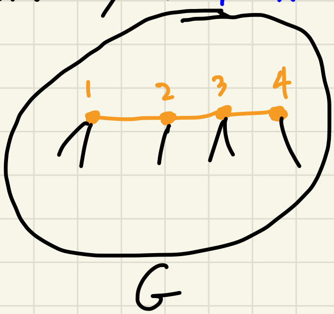
Equation corresponding to vertex v:

$$x_{e_1} + x_{e_2} + x_{e_3} = b_v$$

Theorem (Arkhipov): Let M be the incidence matrix of a connected graph G, and let $b \in \mathbb{Z}_2^{V(G)}$. Then

1) $Mx=b$ has a solution $\Leftrightarrow$ b has even weight;
2) if b has odd weight, then $Mx=b$ has a quantum solution $\Leftrightarrow$ G is not planar.

Lemma (Arkhipov): For G & M as above & $b, b' \in \mathbb{Z}_2^{V(G)}$ s.t. $wt(b) \equiv wt(b') \bmod 2$, $Mx=b$ has a classical/quantum solution if & only if $Mx=b'$ has a classical/quantum solution.



$$b \mapsto b + e_1 + e_4$$

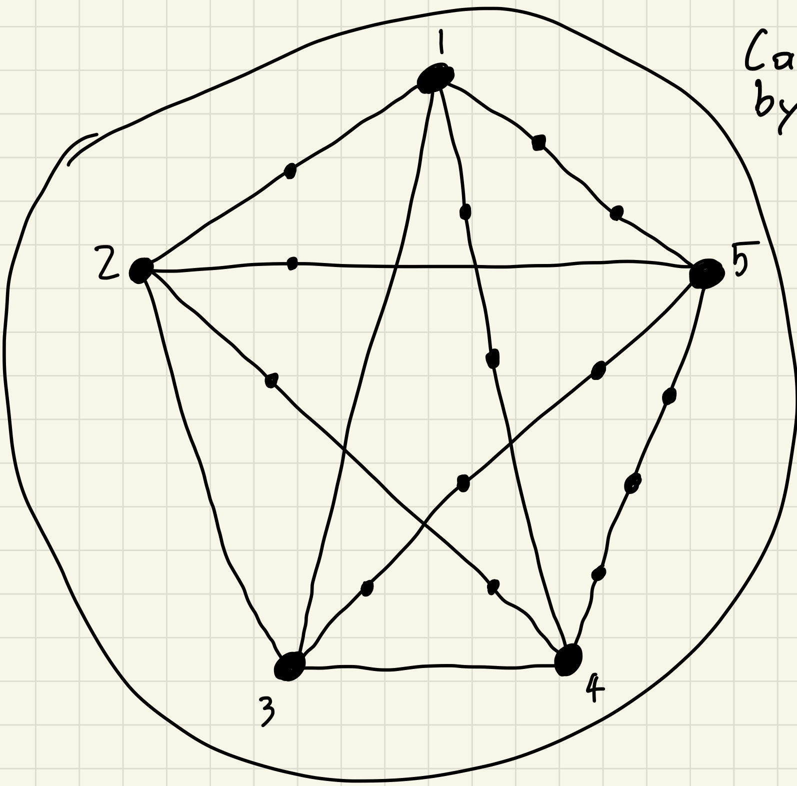
$$x_{ij} \mapsto \begin{cases} x_{ij}+1 & \text{if } ij \in \{12, 23, 34\} \\ x_{ij} & \text{o.w.} \end{cases}$$

Theorem proof:

(1): $x \in \mathbb{Z}_2^{E(G)}$ satisfies $Mx=b$ if and only if the graph $(V(G), \{e \in E(G) : x_e = 1\})$ satisfies

$$\deg(v) \equiv b_v \mod 2.$$

(2): $(\Leftarrow)$ G not planar $\Rightarrow$ G has a $\underbrace{K_{3,3} \text{ or } K_5}_{Mx=b \text{ has q-solution}}$ subdivision:

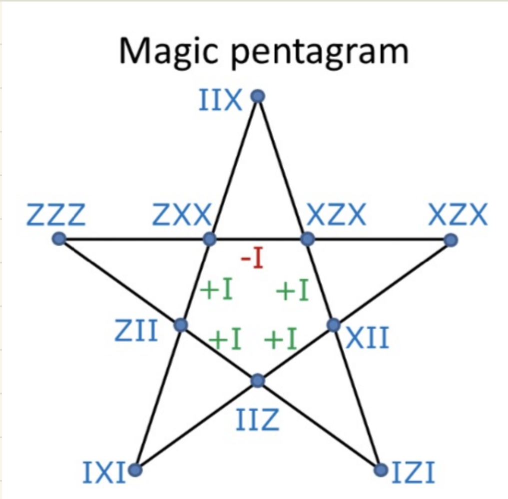


Can assume $b = e_1$ by Lemma

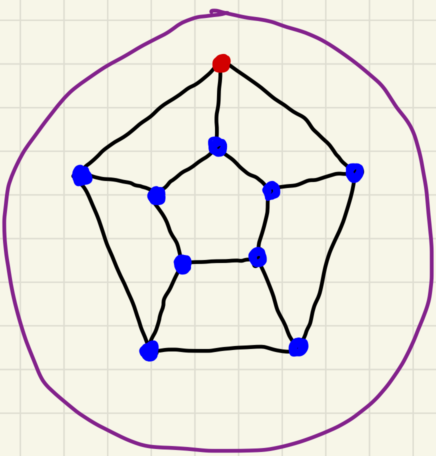
G

$G = K_{3,3}$: Magic square from last lecture.

$G = K_5$:



($\Rightarrow$) By contrapositive. If G is planar then draw G in the plane:



• $b_v = 1$

• $b_v = 0$

No edges incident to boundary $\Rightarrow$ $e = J$.

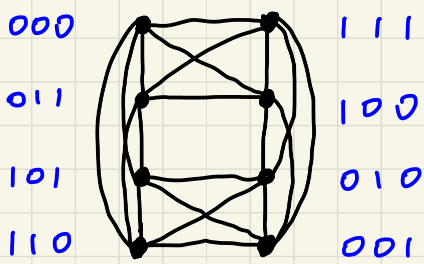
Corollary: If M is the incidence matrix of a connected non-planar graph G and $b \in \mathbb{Z}_2^{V(G)}$ has odd weight, then $Mx=b$ has a quantum solution but no classical solution.

Remark: When M is the incidence matrix of a graph, the system $Mx=b$ has a quantum solution if and only if it has a finite dimensional quantum solution.

A graph associated to $Mx=b$

Let $M \in \mathbb{Z}_2^{m\times n}$ & $b \in \mathbb{Z}_2^m$. The graph $G(M,b)$ has vertex set $\bigcup_{\ell=1}^{m} \{f: S_\ell \to \mathbb{Z}_2 \mid \sum_{i\in S_\ell} f(i) = b_\ell\}$ and $f: S_\ell \to \mathbb{Z}_2$ & $f': S_k \to \mathbb{Z}_2$ are adjacent if $\exists\ i \in S_\ell \cap S_k$ s.t. $f(i) \neq f'(i)$.

Example: $x_1 + x_2 + x_3 = 0$, $\quad x_1 + x_4 + x_6 = 1$



Remark: The sets $\{f: S_\ell \to \mathbb{Z}_2 \mid \sum_{i\in S_\ell} f(i) = b_\ell\}$ for $\ell \in [m]$ partition $V(G(M,b))$ and each such set induces a clique (complete subgraph).

$\Rightarrow \alpha(G(M,b)) \leq \alpha_q(G(M,b)) \leq \alpha_{qc}(G(M,b)) \leq \chi(\overline{G(M,b)}) \leq m$

Atserias, Mančinska, Roberson, Šámal, Severini, & Varvitsiotis

Theorem: For $M \in \mathbb{Z}_2^{m \times n}$ & $b \in \mathbb{Z}_2^m$, the following are equivalent:

1) $Mx = b$ has a solution;
2) $G(M,b) \cong G(M,0)$;
3) $\alpha(G(M,b)) = m$.

Proof: Exercise.

Theorem: For $M \in \mathbb{Z}_2^{m \times n}$ & $b \in \mathbb{Z}_2^m$, the following are equivalent:

1) $Mx = b$ has a finite dimensional quantum solution;
2) $G(M,b) \cong_q G(M,0)$;
3) $\alpha_q(G(M,b)) = m$;
4) $G(M,b)$ has a projective packing of value m.

Theorem: For $M \in \mathbb{Z}_2^{m \times n}$ & $b \in \mathbb{Z}_2^m$, the following are equivalent:

1) $Mx = b$ has a quantum solution;
2) $G(M,b) \cong_{qc} G(M,0)$;
3) $\alpha_{qc}(G(M,b)) = m$;
4) $G(M,b)$ has a tracial packing of value m.

Proof: (1) ⇒ (2) In terms of the BLS & isomorphism games. I.e. we assume Alice & Bob have a strategy for the (M,b)-game, and we use this to produce a strategy for the $(G(M,b), G(M,0))$-iso game.

- Alice receives $f_A : S_\ell \to \mathbb{Z}_2$ s.t. $\sum_{i \in S_\ell} f_A(i) = b_\ell$.
- Acts as if she received ℓ in (M,b)-game to obtain $f_A' : S_\ell \to \mathbb{Z}_2$ s.t. $\sum_{i \in S_\ell} f_A'(i) = b_\ell$.
- Responds w/ $f_A + f_A' : S_\ell \to \mathbb{Z}_2$ defined as $f_A + f_A'(i) = f_A(i) + f_A'(i)$.

Note: $\sum_{i \in S_\ell} f_A + f_A'(i) = \sum_{i \in S_\ell} f_A(i) + \sum_{i \in S_\ell} f_A'(i) = b_\ell + b_\ell = 0$

$\Rightarrow f_A + f_A' \in V(G(M,0))$

- Bob behaves similarly with $f_B, f_B' : S_k \to \mathbb{Z}_2$.

Check iso-game conditions

$f_A = f_B \Rightarrow f_A + f_A' = f_B + f_B'$: $f_A = f_B \Rightarrow \ell = k \Rightarrow f_A' = f_B'$.

$f_A \sim f_B \Rightarrow f_A + f_A' \sim f_B + f_B'$: $f_A \sim f_B \Rightarrow \exists\, i \in S_\ell \cap S_k$ s.t. $f_A(i) \neq f_B(i)$

but $f_A'(i) = f_B'(i) \Rightarrow f_A + f_A'(i) \neq f_B + f_B'(i)$.

$f_A \nsim f_B \Rightarrow f_A + f_A' \nsim f_B + f_B'$: Similar.

(1) $\Rightarrow$ (2) done.

(2) $\Rightarrow$ (3):

$G(M,b) \cong_q G(M,0) \Rightarrow \alpha_q(G(M,b)) = \alpha_q(G(M,0)) = m$

since $m = \alpha(G(M,0)) \leq \alpha_q(G(M,0)) \leq m$

Same for qc.

(3) $\Rightarrow$ (4):

$\alpha_q(G) = k \Rightarrow G$ has a projective packing of value k.

Same for qc.

$(4) \Rightarrow (1)$: Exercise.